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ABSTRACT

Explored were two issues raised by hypothesizing a direct link between number and functions: the relative contribution of distinct types of functions to numerical reasoning and the contribution of functions relative to the contribution of logic. Seventy-two children between the ages of 5 and 7 years were individually tested on measures of arithmetic operations, number conservation, composition of "extensional" and "intensional" functions, morphisms, class inclusion, class vicariance, and seriation. Three versions of each type of problem were administered, for a total of 24 tasks. All problems employed concrete materials. Performances were videotaped and coded on a six-point scale that took into account response content and quality of justifications and/or strategies. Findings support the general contention of the Piagetian model, i.e., that cognition of functions lays the groundwork for concrete operational cognition. In multiple regression analyses, extensional function ability was found to account for 62 percent of the variance in the measures of overall numerical ability. Intensional logic ability was found to be a significant predictor of all logic concepts tested. It is concluded that results suggest a new interpretation of the relation between number, class, and order, as well as a potential amplification of the Piagetian model which would take account of possibly distinct domains of prelogical and prenumerical functions in early thought. (RH)

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The Relation Between the Cognition of Functions and the Construction of Number

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Paper presented at the 14th Annual Symposium of the Jean Piaget Society, Philadelphia, May 31, 1984.

The concept of function is fundamental in all branches of mathematics and in formal logic. As Whitehead (1911/1948, p. 107) pointed out, the concept has also been adopted in ordinary language. "For example, 'His temper is a function of his digestion,' uses the term exactly in this mathematical sense. It means that a rule can be assigned which will tell you what his temper will be when you know how his digestion is working." There has recently been considerable interest in mathematical functions as a model of cognition at various age levels (e.g., Case, 1982; Fischer, 1980; Halford, 1982; Langer, 1980; Piaget, Grize, Szeminska, & Vinh-Bang, 1968/1977). Piaget in particular has proposed that cognition of functional relations in early childhood lays the groundwork for the binary operations of the concrete operational period.

Despite that interest, there has been little inquiry into developmental relations between the function concept and the domain of numerical reasoning. For instance, the Genevan work on functions (Piaget et al., 1968/1977) was primarily directed at providing a positive characterization of logical and causal cognition during early childhood. Piaget traced the cognition of functions to elementary sensorimotor action schemes. During the period of 4 to 6 years, children acquire a representational logic of functions, and this contains the necessary preconditions for concrete operational class and order structures. Piaget, at least in this 1968 model, did not relate the acquisition of number concepts directly to the logic of functions, but continued to view number as a synthesis of class and order operations (Piaget et al., 1968/1977, p. 185).

By algebraic considerations, a close connection should be expected between children's actions involving properties of functions, and acquisition of numerical operations. The prototype of all number systems is the ordered group; that is, a structure consisting of an ordered set of elements, an inverse for each element, an associative binary operation, and an identity. It is a well known theorem of algebra (Cayley's theorem) that every group is isomorphic to a transformation group, that is, to a group having functions as elements and functional composition as its binary operation (functional composition refers to applying two functions in sequence). For instance, the group of integers under addition is interpretable as a group of functions [. . . , $\underline{x} - 2$, $\underline{x} - 1$, $\underline{x} + 0$, $\underline{x} + 1$, $\underline{x} + 2$, . . .] under composition. Number is in fact defined in terms of functions in category-theoretic approaches to mathematical foundations (Lawvere, 1966) and in combinatory logic (Fitch, 1974).

It is conceivable that children's number notions may arise in part from their function notions. For example, young children may initially conceive of a number such as 3 as "make 3 more." This would essentially involve thinking of number as a kind of action (or collecting) rather than a static set of objects (or collection). One theoretical advantage to this interpretation is that it provides a natural explanation for the construction of binary arithmetic operations: these would originate as the sequential composition of two functions (or more precisely, of two unary operations). For instance, "make 3 more" followed by "make 2 more" equals "make 5 more." Such equivalences among action sequences may serve as a conceptual foundation for the acquisition of binary operations (e.g., 3 + 2 = 5).

Current evidence (e.g., Langer, 1980; Forman, 1982) indicates that children begin to construct prequantitative functional relations as early as infancy; hence the suggestion is plausible that number may originate, in part, as an action scheme with function properties. The interpretation of the source of addition and subtraction as a composition of two "collectings" is consistent with suggestions that children more readily conceive of arithmetic operations as unary operations (Weaver, 1982); and with the considerable evidence that children's early addition strategies involve a sequential composition of separate increments (i.e., the "counting-all" strategy; c.f. Fuson, 1982).

My research explores two issues raised by hypothesizing a direct link between number and functions. The first concerns the relative contribution of distinct types of functions to numerical reasoning. For example, I distinguish in this study between "extensional" and "intensional" functions. "Extensional" here refers to functions defined on specific individual objects. As an illustration, one of the function problems in this study involved mapping specific positions of a toy car on a circular road to other positions on the road. The aspect of function reasoning tested was functional composition; in this case composition involved combining two motions of the car in a sequence. "Intensional" refers to functions defined on predicates rather than individual elements. For instance, some problems involved mapping colored objects to other colored objects; the specific object didn't matter as long as the colors were correct. The aspect tested was again functional composition, here represented by combining a sequence of mappings to obtain a color that could not be obtained in a single

mapping. It is important to note that none of the function problems entailed any explicit numerical knowledge. That is, the purpose of these tasks was to assess children's cognition of the most general properties of functions (e.g., functions have unique values; functions can be composed in sequence; etc.). The hypothesis was that such knowledge of function properties would predict children's performance on numerical problems.

The second issue explored in the study concerns the contribution of functions relative to that of logic, of special interest given Piaget's longstanding interpretation of number as a synthesis of logical class and order operations (Piaget, 1941/1965; 1972, Chap. 4; 1980, p. 195). Accordingly, in addition to numerical and function problems, children were tested on tasks involving the logic of classes and relations. It was expected that the association between number and function abilities would be independent of the contribution of these logical abilities.

Subjects and Procedure

Seventy-two 5- through 7-year old children were tested individually on three measures of each of the abilities listed in table 1, for a total of 24 tasks.[1]

Insert table 1 about here

All problems employed concrete materials. Performances were videotaped, and coded reliably on a 6-point scale that took into account both response content and quality of justifications and/or strategies. Table 2 provides a brief summary of coding criteria used.

Insert table 2 about here

The three measures pertaining to each ability listed in table 1 were composited to yield a single score; arithmetic and conservation were also further composited into a single number concept score.

Results

The goals at this stage of the research program were to determine the relative contribution of the distinct types of function abilities tested to numerical reasoning, and to determine the relative contribution of function ability as contrasted with logical ability. These questions were addressed with multiple regression analyses. The results are given in the upper part of table 3.

Insert table 3 about here

Table 3 presents summaries of six different regression analyses. In each, one variable is treated as dependent, with the other performance variables, and age, entered as predictors. The left-hand column indicates which variable is treated as dependent in a given analysis. The second column of the table lists significant predictors. The analysis of primary interest is that in which the dependent variable is "combined number performance"; this is the composite of scores on all arithmetic and number conservation tasks. Extensional function ability accounts for 62% of the variance (i.e., prior to other variables

entering the equation) in this measure of overall numerical ability. Two measures of class logic (class inclusion and vicariance) contribute further (independent) explanatory power, for a total of 72% of the variance accounted for. The concept of logical order, as measured by seriation tasks, is a poor predictor of numerical competence when the effects of other predictors are controlled.

Table 3 also shows the results of similar regression analyses with logic variables treated as dependent. These analyses show that intensional function ability is a significant predictor of all logic concepts tested, but none of the numerical concepts; whereas extensional function ability significantly predicts the numerical concepts but none of the logic concepts.

Conclusions

The findings support the general contention of the Piaget et al. (1968/1977) model, that cognition of functions lays the groundwork for concrete operational cognition. However, the functions associated with number, the extensional functions, are distinct from the intensional functions which are associated with logic. This suggests a potential amplification of the Piagetian model, which would take account of possibly distinct domains of prelogical and prenumerical functions in early thought.

The hypothesis of a direct link between number and the cognition of functions, independent of logical abilities received support. As noted above, the extensional function measure--the strongest predictor of the number concept--does not itself require explicit numerical knowledge. Hence its relation to number is presumably due to some common basis in

cognition rather than to common task content. One possible commonality, mentioned previously, is that functional composition may underlie the construction of binary numerical operations. Another commonality is that number and extensional function tasks involve attention both to extensionality (i.e., to individual elements), and attention to order.

I have mentioned that logical order as measured by seriation tasks is a poor predictor of number performance. However, it is important to notice that the extensional function problems themselves involve considerable attention to order. All functions are based on ordered pairs; that is, the order of the mapping (from argument to value) within each pair is critical. Moreover, functional composition is essentially an ordering of ordered pairs.[2] This is basically a spatio-temporal order, involving a before-after relation, rather than a logical order in the sense of ordered magnitudes such as that defined by Piaget's Grouping V.[3]

Therefore, the present results not only indicate a link between number and functions, but also suggest a new interpretation of the relation between number, class, and order. The findings suggest that it may be useful to describe the construction of number as a coordination of class and function concepts--while emphasizing that order is an intrinsic aspect of the function component. Such a description of psychogenesis is not incompatible with, but rather extends Piaget's analysis of number as a synthesis of class and order concepts.

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Footnotes

[1]The description of task procedures is too extensive to present here in detail. Further information about specific tasks may be obtained by writing the author at the Department of Educational Psychology, College of Education, Michigan State University, East Lansing, MI 48824-1034.

[2]Piaget also stressed the intrinsic order entailed in function concepts, and attributed the precocity of certain order notions during the period of 4 to 6 years to the emergence of the logic of functions (Piaget, 1968; Piaget et al., 1968/1977).

[3]It is interesting that whereas Piaget's formal analyses (e.g., Piaget, 1972) relate number to logical order (Grouping V), his informal discussions refer to infralogical orderings in space and time (e.g., Piaget, 1971, p. 38; Piaget & Inhelder, 1966/1969, p. 105).

Table 1

Summary of Measures Administered

(a) Number:	(i)	arithmetic operations (addition, subtraction, multiplication, division)
	(ii)	number conservation
(b) Functions:	(iii)	composition of extensional functions
	(iv)	composition of intensional functions
	(v)	morphisms
(c) Logic:	(vi)	class inclusion
	(vii)	class vicariance
	(viii)	seriation

Note. Three versions of each type of problem were administered, for a total of 24 tasks.

Table 2

Summary of Scoring Criteria

Level 0	No success; no understanding of how to approach the problem.
Level 1	No success; some understanding of the problem; makes reasonable efforts.
Level 2	Minimal or partial success by groping.
Level 3	Substantial or complete success by groping.
Level 4	Complete success without groping.
Level 5	Complete success with complete certainty (e.g., rejects countersuggestions).

Note. Details of coding varied by task.

Table 3

Summary of Multiple Regression Analyses of Composite Variables

Dependent variable	Variables in the equation	beta	t-test	Cumulative R^2
Number variables				
Arithmetic	Extensional functions	.42	4.67***	.55
	Seriation	.19	2.35*	.62
	Age	.24	2.96**	.66
	Vicariance	.21	2.52**	.69
Conservation	Extensional functions	.43	3.93***	.38
	Class inclusion	.33	3.00**	.45
Combined number performance	Extensional functions	.53	6.30***	.62
	Class inclusion	.25	3.11**	.69
	Vicariance	.21	2.71**	.72
Logic variables				
Class inclusion	Number conservation	.41	4.23**	.40
	Intensional functions	.40	4.18***	.46
Vicariance	Intensional functions	.25	2.03*	.39
	Morphisms	.31	3.05**	.47
	Arithmetic	.29	2.52**	.52
Seriation	Intensional functions	.38	3.09**	.37
	Arithmetic	.34	2.77**	.43

Note. Combined number performance is the average of Arithmetic and Number Conservation scores.

*p < .05

**p < .01

***p < .001